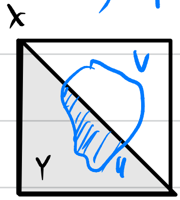
Metric Spaces and Topology
Lecture 4

Relatively open sets. Let (X,d) be a metric space and let $Y \subseteq X$. Then a set $U \subseteq Y$ is called *relatively open with respect to Y* or *open in Y* if U is just an open set in the metric space (Y,d).



Obs. The relatively open sets with respect to Y are exactly the sets $V \cap Y$ for an open set $V \subseteq X$ (open in X).

Proof. We showed that this is true for open balls and open sets are unions of open balls. □

Example. The set $[0, \frac{1}{2})$ is not open in $\mathbb{R}$ but is open relative to $[0,1)$.

Fact. In any metric space (X,d), X and $\emptyset$ are open.

Closed sets. Closed sets are just the complements of open sets.

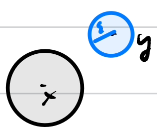
Obs. Arbitrary intersections of closed sets are closed and finite unions of closed sets are closed.

Proof. Follows from the analogous properties of open set by taking complements (de Morgan's laws).

Fact. In any metric space (X,d), X and $\emptyset$ are closed. Thus, they are **clopen** (closed and open).

Examples.

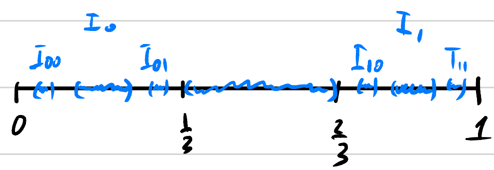
- Closed balls are closed, in any metric sp. (X,d).

  Proof. Let $\overline{B}_r(x) := \{y \in X : d(x,y) \le r\}$.
  Let $y \notin \overline{B}_r(x)$, so $d(x,y) > r$ so
  $\exists\, \varepsilon > 0$ s.t. $d(x,y) > r + \varepsilon$ so
  $B_\varepsilon(y) \cap \overline{B}_r(x) = \emptyset$ by the Δ-ineq. $\square$

- In particular, all closed intervals in $\mathbb{R}$ are closed.

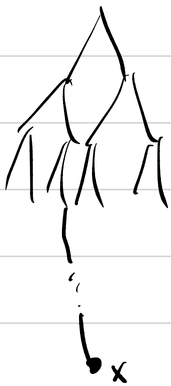
- Cantor set.

  $$C = \underbrace{\bigcap_n \underbrace{\bigcup_{s \in \{0,1\}^n} I_s}_{\text{closed}}}_{\text{closed}}$$

  This is a closed set and

we'll show that there is a natural bijection between C and $2^{\mathbb{N}}$. This bijection is defined as follows:

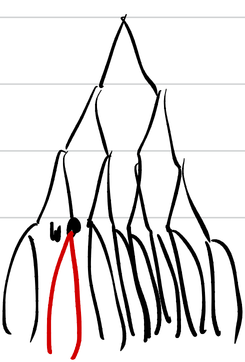


$$f: 2^{\mathbb{N}} \to C$$
$$x \mapsto \bigcap_n I_{x|_n}, \quad \text{where } x|_n := (x_0, x_1, \dots, x_{n-1})$$

By the completeness of $\mathbb{R}$, it follows that $\bigcap_n I_{x|_n} \neq \emptyset$. (We'll prove this later.) This is the lemma of nested intervals.

Remark. This f is a bijection and both f and f^{-1} are continuous. Thus C is homeomorphic to $2^{\mathbb{N}}$.

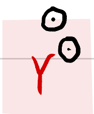
- Consider $2^{\mathbb{N}}$ with the usual metric d.
The open balls here are the same a closed balls, and open balls are the same as cylinders $[w] := \{ w^\frown x : x \in 2^{\mathbb{N}} \}$, thus these sets are clopen.
Indeed, if $|w| = n$, then $[w]^c = \bigcup_{w \neq w' \in 2^n} [w']$.



Recall. The largest open subset of a set $Y \subseteq X$ is the $\mathrm{int}(Y)$.

Def. For a subset Y of a metric space (X,d), we define $\overline{Y}$ as the smallest closed superset of Y. This exists because $\mathrm{int}(Y^c)^c$ fullfils the requirement, $\overline{Y} = \mathrm{int}(Y^c)^c$. This is called the *closure* of Y.

Prop. $\overline{Y}$ = the set of all adherent points to Y, where a point $x \in X$ is said to be *adherent* to Y if any neighbourhood of x intersects Y.



$\Updownarrow$ open ball around x

Proof. This follow from the def of $\mathrm{int}(Y^c)$, indeed,

$$\mathrm{int}(Y^c) = \{y \in X : \exists \varepsilon > 0 \ B_\varepsilon(y) \subseteq Y^c\}, \text{ so}$$

$$\mathrm{int}(Y^c)^c = \{y \in X : \forall \varepsilon > 0 \ B_\varepsilon(y) \cap Y \neq \emptyset\}. \qquad \square$$

Density. A set $D \subseteq X$ in a metric space (X,d) is called *dense* if it has a representative (i.e. intersects) every nonempty open set.

Obs. A set is dense $\Leftrightarrow$ its closure is the whole space.

Examples. ○ In $\mathbb{R}$, the following sets are dense:

- $\mathbb{Q}$
- $\mathbb{R}$
- $\sqrt{2}+\mathbb{Q}$, $\sqrt{2}\cdot\mathbb{Q}$ are dense
- $\mathbb{R}\setminus\mathbb{Q} \supseteq \sqrt{2}+\mathbb{Q}$ so is dense.

∘ In $\mathbb{R}^2$, $D_1 \times D_2$ is dense if both D_1 and D_2 are dense, e.g. $\mathbb{Q}\times(\sqrt{2}+\mathbb{Q})$.

∘ In $\mathbb{N}^{\mathbb{N}}$, the following set is dense:

$$\{w000\ldots : w \in \mathbb{N}^{<\mathbb{N}}\}.$$

Obs. A set $D \subseteq \mathbb{N}^{\mathbb{N}}$ is dense $\Leftrightarrow$ $\forall w \in \mathbb{N}^{<\mathbb{N}}\ \exists x \in \mathbb{N}^{\mathbb{N}}$ s.t. $w^\frown x \in D$.

HW. $\Omega \subseteq \mathbb{N}^{<\mathbb{N}}$ is called *dense* if $\forall w \in \mathbb{N}^{<\mathbb{N}}$ $\exists w' \in \mathbb{N}^{<\mathbb{N}}$ s.t. $w^\frown w' \in \Omega$.

(a) Prove that if Ω is dense then $\Omega 0^\infty := \{w000\ldots : w \in \Omega\}$ is dense in $\mathbb{N}^{\mathbb{N}}$.

(b) Does there exist a dense $\Omega \subseteq \mathbb{N}^{<\mathbb{N}}$ that contains exactly one word of each length?

- $[0,1] \setminus$ Cantor set is an open dense set in $[0,1]$. HW

Limits of sequences. A sequence (x_n) in a set X is just a function $\mathbb{N} \to X$.

Def. Let P be a property of natural numbers. We write $\forall^\infty_n P(n)$ to mean all $n \in \mathbb{N}$ except for finitely many satisfy P. Formally,

$$\forall^\infty_n P(n) \iff \exists N \ \forall n \geq N \ P(n).$$

We write $\exists^\infty_n P(n)$ to mean that infinitely many $n \in \mathbb{N}$ satisfy P. Formally,

$$\exists^\infty_n P(n) \iff \forall N \ \exists n \geq N \ P(n).$$

Obs. $\neg \forall^\infty_n P(n) \iff \exists^\infty_n \neg P(n)$, where $\neg$ denotes negat.

Examples. $\exists^\infty_n$ (n is prime), $\forall^\infty_n \left(2^n \geq 7n^{100}\right)$.

Def. In a metric space (X,d), a sequence (x_n) is said to **converge** to $x \in X$ if $\forall$ neighbourhood U of x, $\forall^\infty_n \ x_n \in U$.